Int. J. Solids Structures, 1972, Vol. 8, pp. 1353 to 1372. Pergamon Press. Printed in Great Britain

DYNAMIC ANALYSIS OF AN INERTIAL FOUNDATION MODEL

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Abstract—The three-parameter weightless foundation model proposed by Kerr [29] which consists of two elastic spring layers interconnected by an elastic shear layer, is generalised herein to include the effect of foundation inertia. The steady-state motion of a rigid beam resting on the surface of a two-dimensional model is analysed considering permanent contact and a distributed mass of the shear layer. A large set of patterns are given for the contact pressure distribution under the beam, pointing out its variation with the exciting frequency in both phase and magnitude. Qualitative agreement with the available experimental data is found for a limit case.

NOTATION

C_1, C_2, C_3, C_4	integration constants in equations (10) and (11)
F_0	amplitude of harmonic force acting on beam, Fig. 1
G	shear layer constant
J_b	mass moment of inertia of the beam
M_0	amplitude of harmonic couple acting on beam (anti-symmetric case)
j	$= \sqrt{-1}$
k_1, k_2	upper and lower spring layer constants, respectively
l	half-length of the beam
m	mass per unit length of the shear layer
n	dimensionless foundation parameter, equation (9)
q	amplitude of the external pressure applied to the foundation surface
t	time
v_1, v_2, y_1, y_2	deflections of the foundation surface and shear layer, respectively
v_0	amplitude of the vertical displacement of the beam
v_{st}^*	static deflection of beam on Winkler foundation (symmetric case)
x	distance from origin
α	notation according to Table 1
β	$= -\bar{\beta}$, notation according to equation (9).
γ_2	dimensionless foundation parameter, equation (9)
$\gamma, \bar{\gamma}, \hat{\gamma}$	abbreviations according to equations (19), (22), (29), respectively
ξ	dimensionless co-ordinate, equation (9)
$\psi, \bar{\psi}$	mass ratios, according to equations (9) and (46)
φ_0	amplitude of rotation angle of the beam (anti-symmetric case)
φ_{st}^*	static rotation angle of the beam on Winkler foundation
μ	mass of the beam
ω	forcing frequency
$\Omega, \bar{\Omega}$	dimensionless forcing frequencies, equations (9) and (46).

1. INTRODUCTION

THE vibration of a rigid body resting on a deformable medium is the basic problem of foundation dynamics. Generally, the supporting medium is replaced by a *foundation model* and some assumptions are made concerning the contact conditions.

The most used model for the theoretical investigation of the footing vibrations is the elastic isotropic continuum, for which linear stress–strain relations are assumed valid. The mixed boundary values problem of the oscillations of a rigid body on an infinite *half-space* was formulated as a set of dual integral equations, corresponding to the uniform displacement conditions under the rigid body and zero stress condition on the surface of the half-space outside the body [1]. For uncoupled modes of vibration and smooth contact between body and foundation accurate solutions have been proposed by Awojobi and Grootenhuis [2], Robertson [3], Karasudhi *et al.* [4]. A survey of the available literature can be found in a paper by Gladwell [5]. Approximate solutions had been published earlier by Reissner [6], Sung [7] and Bycroft [8] by assuming a certain distribution of stresses over the contact area or by averaging the computed displacements of the contact area. Of real interest is Lysmer's numerical "ring solution" [9] based on Ref. [6].

The vertical vibrations of a rigid body on an *elastic stratum* resting on a rigid base have been investigated by Arnold *et al.* [10], Bycroft [8] and Warburton [11]. The dynamic displacement was calculated (using the solution for the half-space) as a weighted average of the values over the whole loaded area. Bycroft's results suggest the possibility of using a model consisting of a set of laterally constrained rods, free at the top, fixed at the base, the length of which equals the depth of stratum (Korenev [12], Viksne [13], Nikolaenko [14]). A new approach to the problem was given by Whitman [15, 16] pointing out the usefulness of "lumped" representation in practical calculations.

In a recent paper by Nowak [17] the amplitudes of the vertical vibrations of massive footings are found to be considerably larger than the values predicted by the half-space theory, so that it is concluded that the soil behaves rather like an elastic stratum with reflections of elastic waves. Adding the nonlinearity, the anisotropy and the general rheological properties of the supporting media it is obvious that the model of the elastic continuum is no cure-all (Whitman [18]). The same can be said for the completely discontinuous and anisotropic Winkler model [19], very often used in engineering practice, consisting of a system of massless linear independent springs which offer resistance in the direction of their axes only.

It is generally accepted that, owing to the extremely various physical nature of foundation materials, a single model cannot describe the mechanical behaviour of all types of subgrade. Starting from the isotropic continuum and introducing simplifying assumptions with respect to the displacements and the stresses (Vlasov and Leont'ev [20], Reissner [21], Muravskii [22]) or improving Winkler's model by adding shear interactions among the elastic elements (Filonenko-Borodich [23], Pasternak [24], Hetényi [25]) numerous mathematical and mechanical models have been proposed to describe the response of a deformable layer of finite depth at the contact surface. These continuous foundation models are very suitable for nonlinear and/or viscoelastic problems (see Kerr [26]) and also for more complicated problems involving deformable structures supported by deformable layers [27]. The inertial models derived from the "discrete-element models" formed following the second development tendency are useful in problems where the subgrade inertia cannot be neglected [28]. The development of this kind of foundation model is necessary in order to fill the existing gap between the elastic continuum model and Winkler's model. The approach permits to solve the correct mixed boundary values problem, starting from a given "constitutive equation" (connecting the applied normal pressure and the foundation surface displacements) and a number of specified boundary conditions.

Herein, a two-dimensional Kerr-type [29] inertial model is examined, consisting of two massless spring layers interconnected by an incompressible shear layer (which deforms by transverse shear only) for which a distributed mass is considered (Fig. 1). It is a four-parameter model, defined by one inertial and three elastic parameters. The steady-state

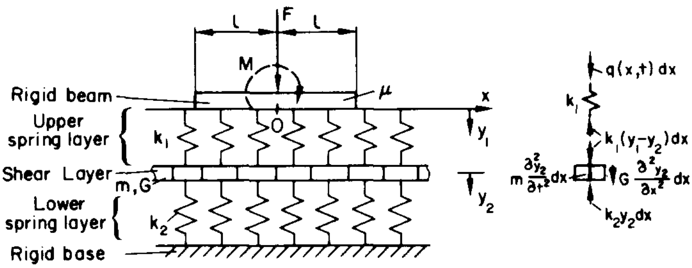

FIG. 1. Rigid beam on an interial foundation with three elastic parameters.

response of a finite rigid beam resting on such a foundation is studied in the assumption of a permanent and smooth contact between beam and foundation. Owing to the anisotropy of the model—which permits only vertical deflections—only vertical translation and rocking can be considered. In the paper only uncoupled modes are considered. Neglect of longitudinal inertia limits the applicability of the proposed model which could however be useful in the case of studded or ribbed carpets used in vibration isolation. A consideration of damping can be achieved by using complex foundation moduli. The dynamic analysis of the corresponding three-dimensional model supporting a rigid circular disk and the experimental determination of foundation constants are the subject of another paper.

2. SYMMETRIC VIBRATIONS

2.1 *Governing equations*

The dynamic equilibrium conditions of the upper spring layer and of the inertial shear layer (Fig. 1) can be written as

$$q(x,t) = k_1(y_1-y_2) = k_2y_2 - G\frac{\partial^2 y_2}{\partial x^2} + m\frac{\partial^2 y_2}{\partial t^2} \tag{1}$$

where k_1 and k_2 are the upper and inner spring layer constants, respectively, G is the shear layer constant, m is the uniformly distributed mass (per unit length) of the shear layer, $q(x, t)$ is the normal pressure at the foundation surface, $y(x, t)$ are vertical deflections to which subscripts 1 and 2 are added for the foundation surface and the shear layer, respectively.

Elimination of y_2 in (1) yields the relationship between the applied pressure and the foundation surface deflection:

$$\left(1+\frac{k_2}{k_1}\right)q - \frac{G}{k_1}\frac{\partial^2 q}{\partial x^2} + \frac{m}{k_1}\frac{\partial^2 q}{\partial t^2} = k_2y_1 - G\frac{\partial^2 y_1}{\partial x^2} + m\frac{\partial^2 y_1}{\partial t^2} \tag{1a}$$

which for $m = 0$ reduces to the equation derived by Kerr [26, 29] (similar equations are presented in [20] and [21]).

Consider a rigid uniform beam, of length $2l$ and mass μ, supported by such a foundation and acted upon in the middle by a harmonic force $F = F_0\,\mathrm{e}^{j\omega t}$, of constant amplitude F_0 and of exciting frequency ω (Fig. 1). Designating the foundation displacements inside and outside the beam by subscripts i and e, respectively, the mixed boundary conditions at the foundation surface are: (1) for $|x| \leq l$, $y_{1i}(x, t) = y_{1i}(t)$; (2) for $|x| > l$, $q(x, t) = 0$, $y_{1e}(x, t) = y_{2e}(x, t)$, so that equation (1) yields

for $|x| \leq l$,

$$k_1 y_{1i} = (k_1 + k_2)y_{2i} - G\frac{\partial^2 y_{2i}}{\partial x^2} + m\frac{\partial^2 y_{2i}}{\partial t^2}; \tag{2}$$

for $|x| > l$,

$$0 = k_2 y_{2e} - G\frac{\partial^2 y_{2e}}{\partial x^2} + m\frac{\partial^2 y_{2e}}{\partial t^2}. \tag{3}$$

For symmetric vibrations, the dynamic equilibrium of the beam requires

$$2\int_0^l k_1(y_{1i} - y_{2i})\,\mathrm{d}x + \mu\frac{\partial^2 y_{1i}}{\partial t^2} = F. \tag{4}$$

Trying solutions of the form

$$y_1(x, t) = v_1(x)\,\mathrm{e}^{j\omega t}, \qquad y_2(x, t) = v_2(x)\,\mathrm{e}^{j\omega t}, \tag{5}$$

the equations (2)–(4) can be written

$$\frac{\mathrm{d}^2 v_{2i}}{\mathrm{d}\xi^2} - \gamma_2^2(n + \beta)v_{2i} = -\gamma_2^2 n v_{1i}, \qquad \text{for } |\xi| \leq 1, \tag{6}$$

$$\frac{\mathrm{d}^2 v_{2e}}{\mathrm{d}\xi^2} - \gamma_2^2 \beta v_{2e} = 0, \qquad \text{for } |\xi| > 1, \tag{7}$$

$$(2k_1 l - \mu\omega^2)v_{1i} - 2k_1 l\int_0^1 v_{2i}(\xi)\,\mathrm{d}\xi = F_0 \qquad \text{for } |\xi| \leq 1, \tag{8}$$

in which the following notations are used

$$\left.\begin{aligned} \xi &= \frac{x}{l}, \qquad \gamma_2 = \left(\frac{k_2 l^2}{G}\right)^{1/2}, \qquad n = \frac{k_1}{k_2}, \\ \beta &= 1 - \frac{1}{\psi}\Omega^2, \qquad \psi = \frac{\mu}{2lm}, \qquad \Omega = \frac{\omega}{\omega_0}, \qquad \omega_0 = \left(\frac{2k_2 l}{\mu}\right)^{1/2} \end{aligned}\right\} \tag{9}$$

where ω_0 can be recognised as being the natural frequency of the beam resting on a Winkler-type foundation of parameter k_2.

2.2 *Solutions for displacements*

The general solutions of the equations (6) and (7) can be written in the form

$$v_{2i}(\xi) = C_1\mathrm{e}^{\gamma\xi} + C_2\mathrm{e}^{-\gamma\xi} + \frac{n}{n + \beta}v_{1i}, \tag{10}$$

$$v_{2e}(\xi) = C_3\mathrm{e}^{\alpha\xi} + C_4\mathrm{e}^{-\alpha\xi}, \tag{11}$$

where γ and α are defined by expressions whose form depends on the sign of β and $(n+\beta)$, as shown in Table 1.

TABLE 1

Case	Frequency range	Condition	γ	α	Analytical expression of v_0	$v_{1e}(\xi)$	$v_{2i}(\xi)$	$\bar{q}(\xi)$	Notation
1	$0 < \Omega < \psi^{1/2}$	$0 < \beta < \beta+n$	$\gamma_2\sqrt{(n+\beta)}$	$\gamma_2\sqrt{\beta}$	(13)	(15)	(14)	(39)	
2	$\psi^{1/2} < \Omega < [\psi(1+n)]^{1/2}$	$\beta < 0 < \beta+n$	$\gamma_2\sqrt{(n-\bar{\beta})}$	$j\gamma_2\sqrt{\bar{\beta}}$	(18)	(20)	(21)	(40)	$\bar{\beta} = -\beta$
3	$[\psi(1+n)]^{1/2} < \Omega$	$\beta < \beta+n < 0$	$j\gamma_2\sqrt{(\bar{\beta}-n)}$	$j\gamma_2\sqrt{\bar{\beta}}$	(23)	(24)	(25)	(41)	$\gamma = j\bar{\gamma}$

Therefore, there are three ranges of exciting frequencies for which the dynamic response will take different shapes. The first boundary $\Omega_1 = \psi^{1/2}$ corresponds to $\omega_1 = (k_2/m)^{1/2}$ which is the resonant frequency of the elastically supported inertial shear layer, for zero external pressure at the foundation surface. The second boundary $\Omega_2 = [\psi(1+n)]^{1/2}$ corresponds to $\omega_2 = [(k_1+k_2)/m]^{1/2}$ which is the resonant frequency of the shear layer for zero deflections at the (fixed) foundation surface.

Because of the symmetry with respect to $\xi = 0$, the problem is studied only for $\xi \geq 0$. The integration constants in equations (10) and (11) are to be determined from the compatibility and boundary conditions for the shear layer:

$$[v_{2e}]_{\xi=1} = [v_{2i}]_{\xi=1}, \qquad \left[\frac{dv_{2i}}{d\xi}\right]_{\xi=0} = 0,$$
$$\left[\frac{dv_{2e}}{d\xi}\right]_{\xi=1} = \left[\frac{dv_{2i}}{d\xi}\right]_{\xi=1} \qquad [v_{2e}]_{\xi\to\infty} = \text{finite}. \tag{12}$$

The expressions of the displacements can be derived using the equations (8) and (10)–(12).

Case 1, $0 < \Omega < \psi^{1/2}$. The vertical displacement of the rigid beam, $v_0 = v_{1i}$, is given by

$$\frac{v_0}{v_{st}^*} = \frac{1}{\dfrac{n}{n+\beta}\left[\beta + \dfrac{n}{\gamma\left[\sqrt{\left(\dfrac{n+\beta}{\beta}\right)} + \dfrac{1}{\tanh\gamma}\right]}\right] - \Omega^2}. \tag{13}$$

The deflections of the shear layer, for $0 \leq \xi < 1$, are

$$v_{2i}(\xi) = v_0 \frac{n}{n+\beta}\left[1 - \frac{\cosh\gamma\xi}{\cosh\gamma + \sqrt{\left(\dfrac{n+\beta}{\beta}\right)}\sinh\gamma}\right]. \tag{14}$$

The foundation surface deflections outside the beam ($\xi > 1$) are given by

$$v_{1e}(\xi) = v_{2e}(\xi) = v_0 \frac{n}{n+\beta} \frac{\sinh\gamma}{\sinh\gamma + \sqrt{(\beta/n+\beta)}\cosh\gamma} e^{-\gamma\sqrt{(\beta/n+\beta)}(\xi-1)} \tag{15}$$

where

$$v_{st}^{*} = \frac{F_0}{2lk_2} \tag{16}$$

is the static displacement (zero frequency) of a rigid beam lying on a Winkler-type foundation of parameter k_2.

It can be seen that v_0 and $v_{1e}(\xi)$ are real quantities. All displacements are in-phase with the force, the foundation surface deflections having an exponential decay outside the beam. At the ends of the beam ($\xi = 1$), the foundation deflection has a discontinuity given by

$$\Delta v = v_{1i}(1) - v_{1e}(1) = v_0 - v_{2e}(1) = v_0\left[1 - \frac{n/n+\beta}{1+\sqrt{(\beta/n+\beta)}(1/\tanh\gamma)}\right]. \tag{17}$$

Case 2, $\psi^{1/2} < \Omega < [\psi(1+n)]^{1/2}$. By substitution of $\beta = -\bar{\beta}$ in the equations (13)–(15), the ratio v_0/v_{st}^{*} can be written as

$$\frac{v_0}{v_{st}^{*}} = \frac{\dfrac{1}{\tanh\gamma} - j\sqrt{\dfrac{n-\bar{\beta}}{\bar{\beta}}}}{\dfrac{n^2}{\gamma(n-\bar{\beta})} - \left(\dfrac{n\bar{\beta}}{n-\bar{\beta}} + \Omega^2\right)\dfrac{1}{\tanh\gamma} + j\sqrt{\left(\dfrac{n-\bar{\beta}}{\bar{\beta}}\right)}\left(\dfrac{n\bar{\beta}}{n-\bar{\beta}} + \Omega^2\right)} \tag{18}$$

in which

$$\gamma = \gamma_2\sqrt{n-\bar{\beta}}. \tag{19}$$

Herein v_0 is a complex amplitude, so that the vertical displacement of the beam is out-of-phase with the disturbing force. Equation (15) becomes

$$v_{1e}(\xi) = v_{2e}(\xi) = v_0\frac{n}{n-\bar{\beta}}\,\frac{1}{1+j\sqrt{(\bar{\beta}/n-\bar{\beta})}/(1/\tanh\gamma)}\,e^{-j\gamma\sqrt{(\bar{\beta}/n-\bar{\beta})}(\xi-1)} \tag{20}$$

describing a wave travelling outward the beam with a velocity

$$\frac{\omega l}{\gamma\sqrt{(\bar{\beta}/n-\bar{\beta})}} = \left(\frac{G}{m}\right)^{1/2}\left(1-\frac{k_2}{m\omega^2}\right)^{-1/2}$$

where $(G/m)^{1/2}$ is the wave propagation velocity in the shear layer alone (unsupported). Note that the velocity of these "shear waves" decreases with increasing exciting frequency, from the infinite value at $\Omega_1 = \psi^{1/2}[\omega_1 = (k_2/m)^{1/2}]$ to the limit value $(G/m)^{1/2}$ at $\Omega \to \infty$, which is in contradiction with the usual experimental results.

Using the same substitution, equation (14) yields

$$v_{2i}(\xi) = v_0\frac{n}{n-\bar{\beta}}\left[1 - \frac{\cosh\gamma\xi}{\cosh\gamma - j\sqrt{(n-\bar{\beta}/\bar{\beta})}\sinh\gamma}\right]. \tag{21}$$

Case 3, $[\psi(1+n)]^{1/2} < \Omega$. Denoting

$$\bar{\gamma} = \gamma_2\sqrt{\bar{\beta}-n} \tag{22}$$

the equations (18), (20) and (21) yield the following expressions for the amplitudes of the beam displacement, the foundation surface deflections and the shear layer deflections beneath the beam:

$$\frac{v_0}{v_{st}^*} = \frac{\dfrac{1}{\tan\bar{\gamma}} + j\sqrt{\left(\dfrac{\bar{\beta}-n}{\bar{\beta}}\right)}}{\left(\dfrac{n\bar{\beta}}{\bar{\beta}-n} - \Omega^2\right)\dfrac{1}{\tan\bar{\gamma}} - \dfrac{n^2}{\bar{\gamma}(\bar{\beta}-n)} + j\sqrt{\left(\dfrac{\bar{\beta}-n}{\bar{\beta}}\right)}\left(\dfrac{n\bar{\beta}}{\bar{\beta}-n} - \Omega^2\right)}, \tag{23}$$

$$v_{1e}(\xi) = v_{2e}(\xi) = v_0 \frac{-\dfrac{n}{\bar{\beta}-n}}{1 - j\sqrt{\left(\dfrac{\bar{\beta}}{\bar{\beta}-n}\right)}\dfrac{1}{\tan\bar{\gamma}}} e^{-j\bar{\gamma}\sqrt{(\bar{\beta}/\bar{\beta}-n)}(\xi-1)}, \tag{24}$$

$$v_{2i}(\xi) = -v_0\frac{n}{\bar{\beta}-n}\left(1 - \frac{\cos\bar{\gamma}\xi}{\cos\bar{\gamma} + j\sqrt{\left(\dfrac{\bar{\beta}-n}{\bar{\beta}}\right)}\sin\bar{\gamma}}\right). \tag{25}$$

As for the previous case, the beam displacement is out-of-phase with the exciting force and the foundation surface exhibits shear-type travelling waves, with the same expression (depending upon the disturbing frequency) for the velocity as in the previous case.

The dynamic response of a rigid beam supported by a Pasternak-type inertial foundation can be obtained as a limit case, by substituting $n \to \infty$ ($k_1 \to \infty$) into the equations (13) and (18) [28]. This yields:

for $0 < \Omega < \psi^{1/2}$,

$$\frac{v_{0P}}{v_{st}^*} = \frac{1}{\beta + (1/\gamma_2)\sqrt{\beta - \Omega^2}}, \tag{26}$$

for $\psi^{1/2} < \Omega$,

$$\frac{v_{0P}}{v_{st}^*} = \frac{1}{j(1/\gamma_2)\sqrt{\beta} - (\beta + \Omega^2)}. \tag{27}$$

For $m = 0$, $\beta = 1$ and equation (13) gives the beam response on a Kerr foundation (non-inertial)

$$\frac{v_0}{v_{st}^*} = \frac{1}{\dfrac{n}{n+1}\left\{1 + \dfrac{n}{\hat{\gamma}[\sqrt{(n+1)} + (1/\tanh\hat{\gamma})]}\right\} - \Omega^2} \tag{28}$$

where

$$\hat{\gamma} = \gamma_2\sqrt{n+1}. \tag{29}$$

By substitution of $\beta = 1$ in equation (26), or $n \to \infty$ in equation (28), the response of the rigid beam on a Pasternak non-inertial foundation can be written as

$$\frac{v_0}{v_{st}^*} = \frac{1}{1 + (1/\gamma_2) - \Omega^2} \tag{30}$$

which, for $G = 0(\gamma_2 \to \infty)$ yields the beam response on a Winkler-type foundation

$$\frac{v_0}{v_{st}^*} = \frac{1}{1-\Omega^2}. \tag{31}$$

The above derived expressions were used for plotting the frequency-response curves in Fig. 2, where the modulus of the vertical displacement of the beam (v_0/v_{st}^*) is presented as a function of the dimensionless exciting frequency Ω for five types of foundation models.

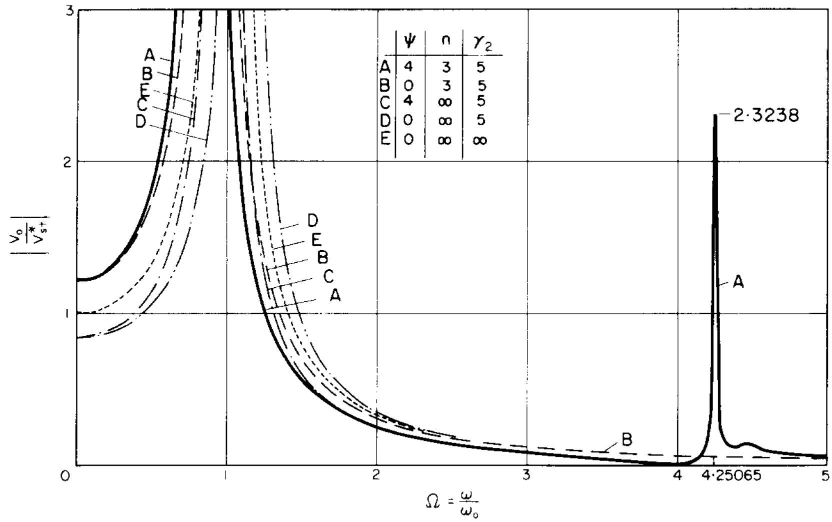

FIG. 2. Frequency-response curves for symmetrical vibrations of a beam on different foundation models.

It can be seen that the Kerr-type inertial foundation model (curve A) exhibits two resonant frequencies. The first one takes place in the frequency range $\Omega < \psi^{1/2}$ and gives rise to boundless amplitudes (in the absence of damping); the second appears in the range $[\psi(1+n)]^{1/2} < \Omega$ and is determined by the inertial shear layer. In this case, the energy dissipation caused by wave propagation, usually referred to as "geometrical damping", limits the resonant amplitudes even in the absence of internal damping.

Neglecting the shear interactions in the four-parameter model ($G = 0$), a Winkler type inertial foundation [Fig. 3(a)] can be obtained, which, for the studied problem, leads to the two-degrees-of-freedom lumped parameter system shown in Fig. 3(b). This corresponds to Whitman's representation [15, 16] for the so-called "soil amplification" and "soil-structure interaction". Figure 4 shows that the beam (mass μ) response in this system (curve B) gives a good approximation for the response on a Kerr inertial foundation (curve A).

In Fig. 5 frequency–response curves are plotted for damped forced vibrations, in the limit case $n \to \infty$ (a Pasternak-type inertial foundation) for $\gamma_2 = 1$. A hysteretic damping is assumed introducing in equation (26) complex foundation parameters

$$k_2^* = k_2(1+j\delta_k), \qquad G^* = G(1+j\delta_G) \tag{32}$$

instead of the real ones (with a rather common assumption $\delta_k = \delta_G = 0$). This results in

$$\frac{v_{0P}}{v_{st}^*} = \frac{1}{(1+j\delta)(1+\gamma'+j\gamma'')-(1+\psi)\Omega_P^2} \tag{33}$$

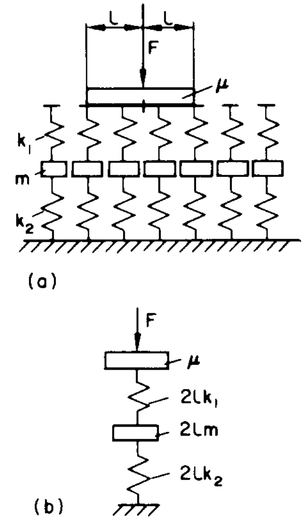

FIG. 3. Two-degree-of-freedom model for an inertial foundation without shear interactions.

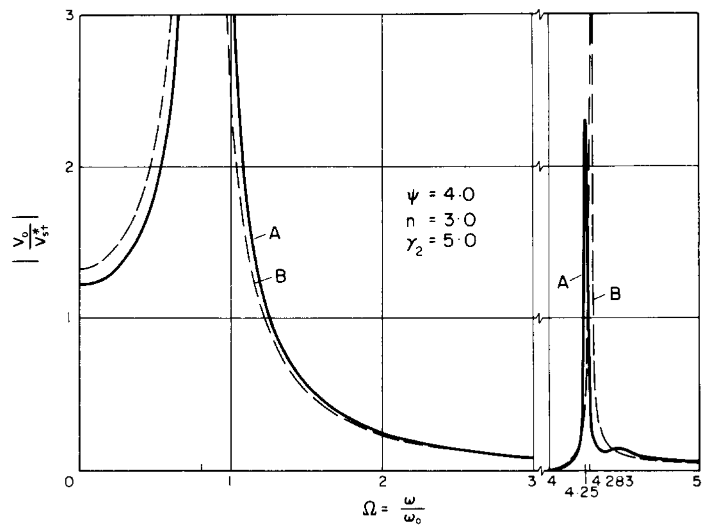

FIG. 4. Displacement amplitude vs. frequency relations for the models in Figs. 1 and 3.

where

$$\Omega_P = \frac{\omega}{\omega_P}, \qquad \omega_P = \left(\frac{k_2}{m}\right)^{1/2}, \tag{34}$$

$$\gamma' = \frac{1}{\gamma_2\Delta\sqrt{2}}\{\Delta^2 - \Omega_P^2 + \sqrt{[(\Delta^2 - \Omega_P^2)^2 + \delta^2\Omega_P^4]}\}^{1/2} \tag{35}$$

$$\gamma'' = -\frac{1}{\gamma_2\Delta\sqrt{2}}\{\Delta^2 - \Omega_P^2 - \sqrt{[(\Delta^2 - \Omega_P^2)^2 + \delta^2\Omega_P^4]}\}^{1/2} \tag{36}$$

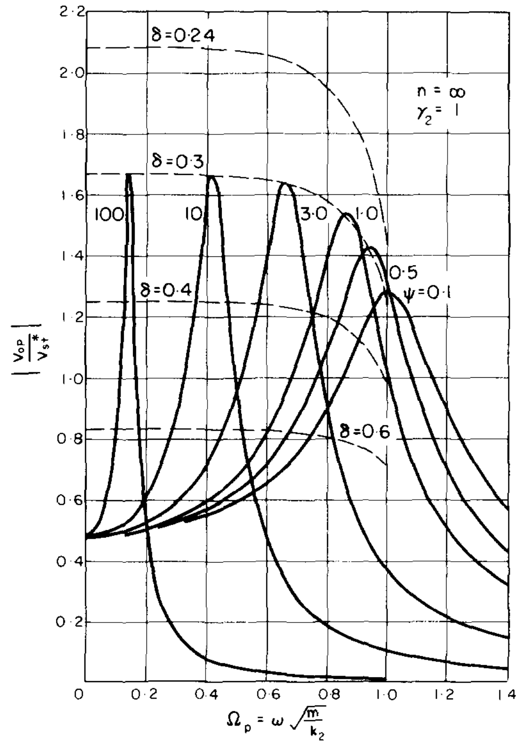

FIG. 5. Frequency-response curves for a beam on an inertial foundation with hysteretic damping.

and

$$\Delta^2 = 1+\delta^2. \tag{37}$$

The peak amplitudes decrease with the mass ratio ψ as shown by broken lines, the slope of which are smaller than for the case of viscous damping [33].

2.4 *Dynamic contact-pressure distribution*

If the beam response can be studied with good results on a two-degree-of-freedom lumped parameter system, conversely, the dynamic pressure distribution and its variation with the forcing frequency can be studied only on a continuous foundation model.

For a steady-state response, $q(\xi, t) = \bar{q}(\xi)\,e^{j\omega t}$, where

$$\bar{q}(\xi) = k_1(v_0 - v_{2i}) \tag{38}$$

which, in connection with equations (13), (14) and (16), yields:

Case 1, $0 < \Omega < \psi^{1/2}$.

$$\frac{\bar{q}(\xi)}{\dfrac{F_0}{2l}} = \frac{\beta\left[\cosh\gamma + \sqrt{\left(\dfrac{n+\beta}{\beta}\right)}\sinh\gamma\right] + n\cosh\gamma\xi}{\left(\beta - \dfrac{n+\beta}{n}\Omega^2\right)\left[\cosh\gamma + \sqrt{\left(\dfrac{n+\beta}{\beta}\right)}\sinh\gamma\right] + \dfrac{n}{\gamma}\sinh\gamma}. \tag{39}$$

Case 2, $\psi^{1/2} < \Omega < [\psi(1+n)]^{1/2}$.

$$\frac{\bar{q}(\xi)}{\frac{F_0}{2l}} = \frac{n\cosh\gamma\xi - \bar{\beta}\cosh\gamma + j\sqrt{[\bar{\beta}(n-\bar{\beta})]}\sinh\gamma}{\frac{n}{\gamma}\sinh\gamma - \left(\bar{\beta}+\frac{n-\bar{\beta}}{n}\Omega^2\right)\cosh\gamma + j\left(\bar{\beta}+\frac{n-\bar{\beta}}{n}\Omega^2\right)\sqrt{\left(\frac{n-\bar{\beta}}{\bar{\beta}}\right)}\sinh\gamma}. \tag{40}$$

Case 3, $[\psi(1+n)]^{1/2} < \Omega$.

$$\frac{\bar{q}(\xi)}{\frac{F_0}{2l}} = \frac{n\cos\bar{\gamma}\xi - \bar{\beta}\cos\bar{\gamma} - j\sqrt{[\bar{\beta}(\bar{\beta}-n)]}\sin\bar{\gamma}}{\frac{n}{\bar{\gamma}}\sin\bar{\gamma} - \left(\bar{\beta}-\frac{\bar{\beta}-n}{n}\Omega^2\right)\cos\bar{\gamma} - j\left(\bar{\beta}-\frac{\bar{\beta}-n}{n}\Omega^2\right)\sqrt{\left(\frac{\bar{\beta}-n}{\bar{\beta}}\right)}\sin\bar{\gamma}}. \tag{41}$$

Substitution of $\beta = 1$ and $\Omega = 0$ into equation (39) gives the contact pressure distribution under a statically and symmetrically loaded rigid beam, resting on a Kerr-type foundation [29]. Figures 6–8 point out the frequency dependence of the pressure distribution under the beam.

In Case 1, the expression (39) defines a real quantity. The contact pressure is in-phase with the perturbation, the distribution is relatively constant in the central part of the beam and increases like "cosh" to a *finite* value at the edges. The peaks are more pronounced for exciting frequencies in the neighbourhood of the bouncing frequency where the usual change of sign takes place (Fig. 6).

In Case 2, the expression (40) defines a complex quantity; the contact pressure can be considered as having two components: $\bar{q}_r$—in-phase with the exciting force and approaching

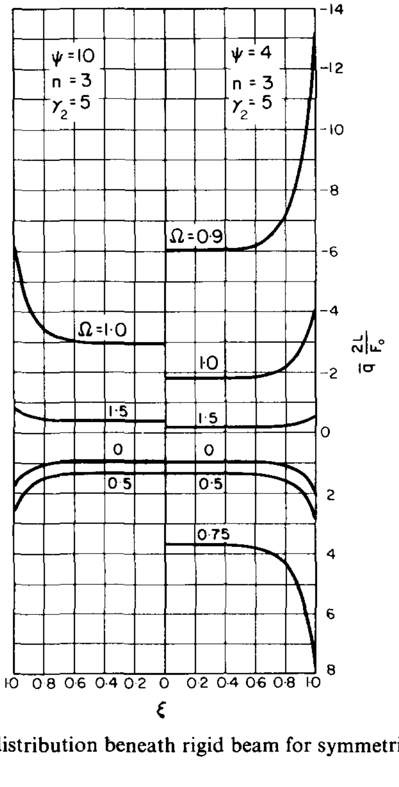

FIG. 6. Contact pressure distribution beneath rigid beam for symmetrical vibrations ($\Omega < \psi^{1/2}$).

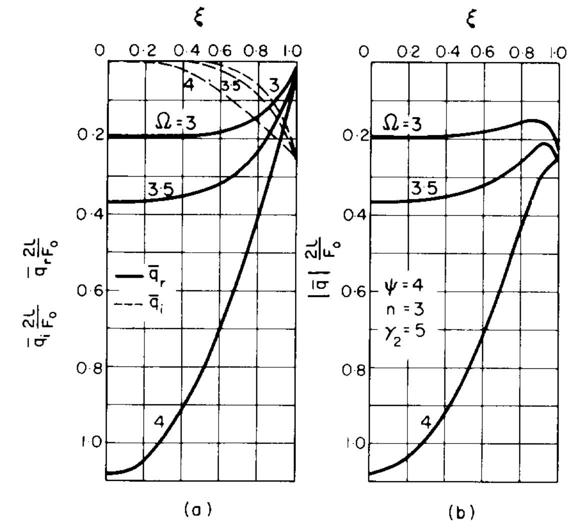

FIG. 7. Contact pressure distribution beneath rigid beam for symmetrical vibrations ($\psi^{1/2} < \Omega < [\psi(1+n)]^{1/2}$).

a parabolic distribution with increasing Ω, $\bar{q}_i$—in quadrature with the exciting force and having a rather "rigid body"-type variation [Fig. 7(a)]. The distribution varies along the beam in both phase and magnitude. In Fig. 7(b) the modulus $\bar{q}/(F_0/2l)$ is plotted vs. ξ for different values of the dimensionless exciting frequency Ω. In this range of frequencies

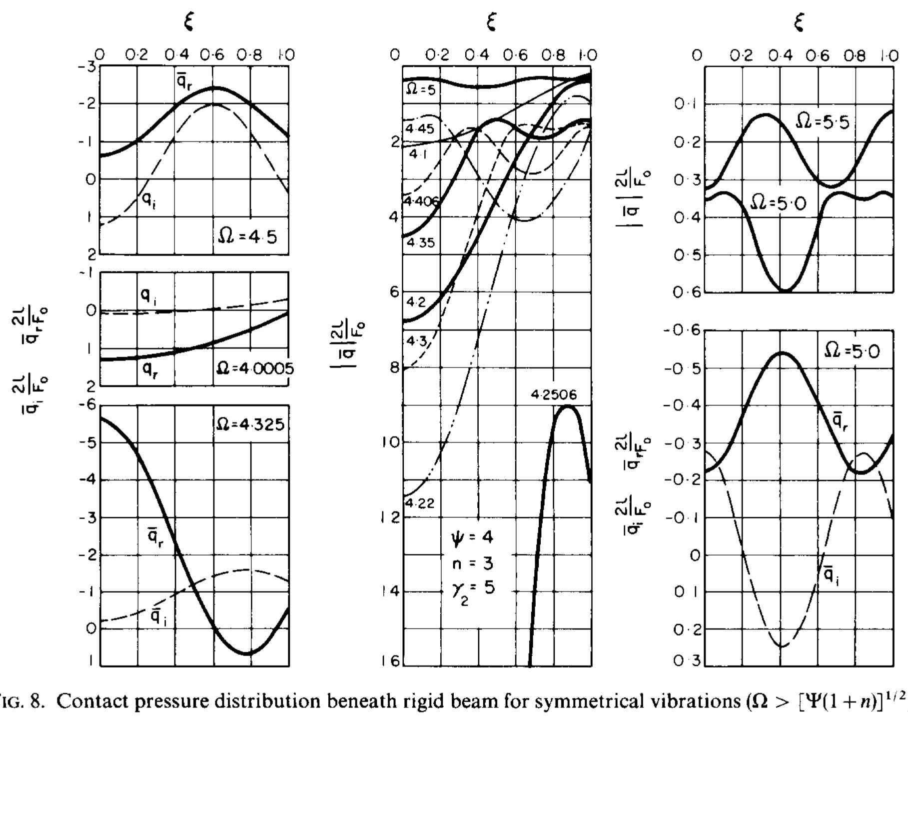

FIG. 8. Contact pressure distribution beneath rigid beam for symmetrical vibrations ($\Omega > [\Psi(1+n)]^{1/2}$).

the contact pressure is relatively small (cf. Figs. 6 and 8) and so is the beam response (see Fig. 2).

In Case 3, for $\Omega > [\psi(1+n)]^{1/2}$, the contact pressure is still out-of-phase with the perturbation, but the distribution has a "cosine"-type variation along the beam, with smaller and mainly negative components beyond the second resonant frequency and approaching a uniform distribution with increasing Ω (Fig. 8).

3. ANTI-SYMMETRICAL VIBRATIONS

3.1 *Equations of motion*

The problem can be treated in a very similar manner as for the symmetric case. Consider a rigid uniform beam of length $2l$, supported by a Kerr-type inertial foundation and acted upon in the middle by a harmonic couple $M = M_0 e^{j\omega t}$ of constant amplitude.

The governing equations (2) and (3), and the upper boundary condition for $|x| > l$ are still valid. The displacement condition beneath beam ($|x| \leq l$) can be written $y_{1i}(x, t) = \varphi(t) \cdot x$, where the rotation angle φ is relatively small. Equation (4) has to be replaced by the equilibrium equation with respect to the beam midpoint

$$2\int_0^l q(x,t)\cdot x \cdot \mathrm{d}x + J_b\frac{\mathrm{d}^2\psi}{\mathrm{d}t^2} = M, \tag{42}$$

where J_b is the mass moment of inertia of the beam.

With solutions of the form (5) and putting $\varphi(t) = \varphi_0 e^{j\omega t}$, the following set of equations of motion can be written:

$$\frac{\mathrm{d}^2 v_{2i}}{\mathrm{d}\xi^2} - \gamma_2^2(n+\beta)v_{2i} = -\gamma_2^2 nl\varphi_0\xi, \qquad |\xi| \leq 1, \tag{43}$$

$$\frac{\mathrm{d}^2 v_{2e}}{\mathrm{d}\xi^2} - \gamma_2^2\beta v_{2e} = 0, \qquad |\xi| > 1, \tag{44}$$

$$\left(\frac{2k_1 l^3}{3} - J_b\omega^2\right)\varphi_0 - 2k_1 l^2\int_0^1 v_{2i}\xi\,\mathrm{d}\xi = M_0, \qquad |\xi| \leq 1, \tag{45}$$

where

$$\beta = 1 - \frac{\bar{\Omega}^2}{\bar{\psi}}, \qquad \bar{\psi} = \frac{3J_b}{2ml^3}, \qquad \bar{\Omega} = \frac{\omega}{\bar{\omega}_0}, \qquad \bar{\omega}_0 = \left(\frac{2k_2 l^3}{3J_b}\right)^{1/2}. \tag{46}$$

3.2 *Solutions for displacements*

The general solutions of the equations (43) and (44) are given by (10) and (11), in which $v_{1i} = l\varphi_0\xi$. The integration constants can be determined using the following conditions for the shear layer:

$$\left.\begin{aligned}&[v_{2e}]_{\xi=1} = [v_{2i}]_{\xi=1}, \qquad [v_{2i}]_{\xi=0} = 0,\\ &\left[\frac{\mathrm{d}v_{2e}}{\mathrm{d}\xi}\right]_{\xi=1} = \left[\frac{\mathrm{d}v_{2i}}{\mathrm{d}\xi}\right]_{\xi=1}, \qquad [v_{2e}]_{\xi\to\infty} = \text{finite.}\end{aligned}\right\} \tag{47}$$

By using the equations (10), (11), (45) and (47), the beam and foundation displacements are obtained as follows:

Case 1, $0 < \bar{\Omega} < \bar{\psi}^{1/2}$. The beam rotation

$$\frac{\varphi_0}{\varphi_{st}^*} = \frac{1}{\dfrac{n}{n+\beta}\left\{\beta + \dfrac{3n\left[\gamma\sqrt{\left(\dfrac{\beta}{n+\beta}\right)}+1\right](\gamma-\tanh\gamma)}{\gamma^3\left[\sqrt{\left(\dfrac{\beta}{n+\beta}\right)}\tanh\gamma+1\right]}\right\}-\bar{\Omega}^2} \tag{48}$$

where $\varphi_{st}^* = 3M_0/2k_2l^3$ is the rotation angle for the beam statically loaded by the couple M_0 and supported by a Winkler foundation of parameter k_2.

The shear layer deflection, for $|\xi| \leq 1$:

$$v_{2i}(\xi) = \varphi_0 l\frac{n}{n+\beta}\left\{\xi - \frac{\left[\gamma\sqrt{\left(\dfrac{\beta}{n+\beta}\right)}+1\right]\sinh\gamma\xi}{\gamma\left[\sqrt{\left(\dfrac{\beta}{n+\beta}\right)}\sinh\gamma+\cosh\gamma\right]}\right\}. \tag{49}$$

The foundation free surface deflections

$$v_{1e}(\xi) = v_{2e}(\xi) = \varphi_0 l\frac{n}{n+\beta}\frac{\dfrac{1}{\gamma}\tanh\gamma-1}{\sqrt{\left(\dfrac{\beta}{n+\beta}\right)}\tanh\gamma+1}\mathrm{e}^{-\gamma\sqrt{(\beta/n+\beta)}(\xi-1)} \tag{50}$$

Case 2, $\bar{\psi}^{1/2} < \Omega < [\bar{\psi}(1+n)]^{1/2}$.

$$\frac{\varphi_0}{\varphi_{st}^*} = \frac{\gamma + j\gamma\sqrt{\left(\dfrac{\bar{\beta}}{n-\bar{\beta}}\right)}\tanh\gamma}{\dfrac{3n^2(\gamma-\tanh\gamma)}{\gamma^2(n-\bar{\beta})} - \left(\bar{\Omega}^2+\dfrac{n\bar{\beta}}{n-\bar{\beta}}\right)\gamma + j\left[\dfrac{3n^2(\gamma-\tanh\gamma)}{\gamma(n-\bar{\beta})}\sqrt{\left(\dfrac{\bar{\beta}}{n-\bar{\beta}}\right)} - \left(\bar{\Omega}^2+\dfrac{n\bar{\beta}}{n-\bar{\beta}}\right)\gamma\sqrt{\left(\dfrac{\bar{\beta}}{n-\bar{\beta}}\right)}\tanh\gamma\right]} \tag{51}$$

$$v_{2i}(\xi) = \varphi_0 l\frac{n}{n-\bar{\beta}}\left\{\xi - \frac{\left[1+j\gamma\sqrt{\left(\dfrac{\bar{\beta}}{n-\bar{\beta}}\right)}\right]\sinh\gamma\xi}{\gamma\cosh\gamma + j\gamma\sqrt{\left(\dfrac{\bar{\beta}}{n-\bar{\beta}}\right)}\sinh\gamma}\right\}, \tag{52}$$

$$v_{1e}(\xi) = v_{2e}(\xi) = \varphi_0 l\frac{n}{n-\bar{\beta}}\frac{\dfrac{1}{\gamma}\tanh\gamma-1}{1+j\sqrt{\left(\dfrac{\bar{\beta}}{n-\bar{\beta}}\right)}\tanh\gamma}\mathrm{e}^{-j\gamma\sqrt{(\bar{\beta}/n-\bar{\beta})}(\xi-1)}. \tag{53}$$

Case 3, $[\bar{\psi}(1+n)]^{1/2} < \bar{\Omega}$.

$$\frac{\varphi_0}{\varphi_{st}^*} = \frac{\sqrt{\left(\dfrac{\bar{\beta}}{\bar{\beta}-n}\right)}\tan\bar{\gamma} - j}{\left(\dfrac{n\bar{\beta}}{\bar{\beta}-n}-\bar{\Omega}^2\right)\sqrt{\left(\dfrac{\bar{\beta}}{\bar{\beta}-n}\right)}\tan\bar{\gamma} + \dfrac{3n^2(\bar{\gamma}-\tan\bar{\gamma})}{\bar{\gamma}^2(\bar{\beta}-n)}\sqrt{\left(\dfrac{\bar{\beta}}{n-\bar{\beta}}\right)} - j\left[\left(\dfrac{n\bar{\beta}}{\bar{\beta}-n}-\bar{\Omega}^2\right) - \dfrac{3n^2(\bar{\gamma}-\tan\bar{\gamma})}{\bar{\gamma}^3(\bar{\beta}-n)}\right]} \tag{54}$$

$$v_{2i}(\xi) = -\varphi_0 l\frac{n}{\bar{\beta}-n}\left\{\xi - \frac{\left[\bar{\gamma}\sqrt{\left(\dfrac{\bar{\beta}}{\bar{\beta}-n}\right)} - j\right]\sin\bar{\gamma}\xi}{\bar{\gamma}\sqrt{\left(\dfrac{\bar{\beta}}{\bar{\beta}-n}\right)}\sin\bar{\gamma} - j\bar{\gamma}\cos\bar{\gamma}}\right\}, \tag{55}$$

$$v_{1e}(\xi) = v_{2e}(\xi) = -\varphi_0 l\frac{n}{\bar{\beta}-n}\,\frac{\tan\bar{\gamma}-\bar{\gamma}}{\bar{\gamma}+j\bar{\gamma}\tan\bar{\gamma}\sqrt{\dfrac{\bar{\beta}}{\bar{\beta}-n}}}\,e^{-j\bar{\gamma}\sqrt{(\bar{\beta}/\bar{\beta}-n)}(\xi-1)}. \tag{56}$$

For $\bar{\Omega} < \bar{\psi}^{1/2}$ the displacements are in-phase with the disturbing couple. For $\bar{\Omega} > \bar{\psi}^{1/2}$ the beam rotation is out-of-phase with the couple and the foundation free surface exhibits shear-type travelling waves having the same frequency-dependent velocity as in the symmetric case. In Fig. 9 the modulus $|\varphi_0/\varphi_{st}^*|$ is plotted against the dimensionless frequency $\bar{\Omega}$, for $n = 3$, $\bar{\psi} = 4$ and $\gamma_2 = 5$. The system has two resonant frequencies, one—in the range $\bar{\Omega} < \bar{\psi}^{1/2}$, of infinite (rotation) amplitude, the other—of finite amplitude and less extent, due to the energy radiation, at relatively high exciting frequencies $\bar{\Omega} > [\bar{\psi}(1+n)]^{1/2}$.

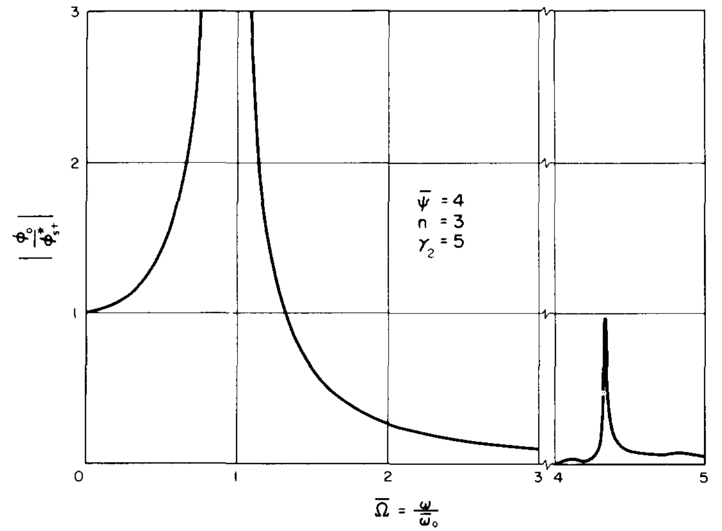

FIG. 9. Frequency–response curves for anti-symmetrical vibrations.

Letting $\beta = 1$ in the equation (48), the beam response on a non-inertial Kerr foundation is obtained as

$$\frac{\varphi_0}{\varphi_{st}^*} = \frac{1}{\dfrac{n}{n+1}\left\{1+\dfrac{3n[\hat{\gamma}+\sqrt{(n+1)}](\hat{\gamma}-\tanh\hat{\gamma})}{\hat{\gamma}^3[\tanh\hat{\gamma}+\sqrt{(n+1)}]}\right\}-\bar{\Omega}^2} \tag{57}$$

giving a good approximation for the low-frequency response of the studied system at low values of the mass ratio $\bar{\psi}$.

3.3 *Contact-pressure distribution under the beam*

By denoting $M_0 = \mathscr{F}_0 L$, where $\mathscr{F}_0$ has the dimensions of a force and L of a length, the following expressions can be derived for the pressure distribution beneath the beam:

Case 1.

$$\frac{\bar{q}(\xi)}{\dfrac{\mathscr{F}_0}{2l}} = \frac{\dfrac{3L}{l}\gamma^2\left\{\left[\gamma\sqrt{\left(\dfrac{\beta}{n+\beta}\right)}\sinh\gamma+\gamma\cosh\gamma\right]\beta\xi+n\left[\gamma\sqrt{\left(\dfrac{\beta}{n+\beta}\right)}+1\right]\sinh\gamma\xi\right\}}{\gamma^2\left(\beta-\dfrac{n+\beta}{n}\bar{\Omega}^2\right)\left[\gamma\sqrt{\left(\dfrac{\beta}{n+\beta}\right)}\sinh\gamma+\gamma\cosh\gamma\right]+3n\left[\gamma\sqrt{\left(\dfrac{\beta}{n+\beta}\right)}+1\right](\gamma\cosh\gamma-\sinh\gamma)}. \tag{58}$$

Case 2.

$$\frac{\bar{q}(\xi)}{\dfrac{\mathscr{F}_0}{2l}} = \frac{\dfrac{3L}{l}\gamma^2\left\{(n\sinh\gamma\xi-\xi\bar{\beta}\gamma\cosh\gamma)+j\left[n\gamma\sqrt{\left(\dfrac{\bar{\beta}}{n-\bar{\beta}}\right)}\sinh\gamma\xi-\xi\bar{\beta}\gamma\sqrt{\left(\dfrac{\bar{\beta}}{n-\bar{\beta}}\right)}\sinh\gamma\right]\right\}}{\left[3n(\gamma\cosh\gamma-\sinh\gamma)-\left(\bar{\beta}+\dfrac{n-\bar{\beta}}{n}\bar{\Omega}^2\right)\gamma^3\cosh\gamma\right]+j\left[3n\gamma\sqrt{\left(\dfrac{\bar{\beta}}{n-\bar{\beta}}\right)}(\gamma\cosh\gamma-\sinh\gamma)+\left(\bar{\beta}+\dfrac{n-\bar{\beta}}{n}\bar{\Omega}^2\right)\gamma^3\sqrt{\left(\dfrac{\bar{\beta}}{n-\bar{\beta}}\right)}\sinh\gamma\right]}. \tag{59}$$

Case 3.

$$\frac{\bar{q}(\xi)}{\dfrac{\mathscr{F}_0}{2l}} = \frac{-\dfrac{3L}{l}\bar{\gamma}^2\left\{\left[\xi\bar{\beta}\sqrt{\left(\dfrac{\bar{\beta}}{\bar{\beta}-n}\right)}\bar{\gamma}\sin\bar{\gamma}+n\bar{\gamma}\sqrt{\left(\dfrac{\bar{\beta}}{\bar{\beta}-n}\right)}\sin\bar{\gamma}\xi\right]+j(\xi\bar{\beta}\bar{\gamma}\cos\bar{\gamma}-n\sin\bar{\gamma}\xi)\right\}}{\left(\dfrac{\bar{\beta}-n}{n}\bar{\Omega}^2-\bar{\beta}\right)\bar{\gamma}^3\sqrt{\left(\dfrac{\bar{\beta}}{\bar{\beta}-n}\right)}\sin\bar{\gamma}+3n(\bar{\gamma}\cos\bar{\gamma}-\sin\bar{\gamma})+j\left[3n(\bar{\gamma}\cos\bar{\gamma}-\sin\bar{\gamma})-\left(\dfrac{\bar{\beta}-n}{n}\bar{\Omega}^2-\bar{\beta}\right)\bar{\gamma}^3\cos\bar{\gamma}\right]}. \tag{60}$$

Substitution of $\beta = 1$ into equation (58) yields the normal interface pressure for a beam on a non-inertial Kerr foundation:

$$\frac{\bar{q}(\xi)}{\frac{\mathscr{F}_0}{2l}} = \frac{\frac{3L}{l}\bar{\gamma}^2\left[\left(\frac{\hat{\gamma}}{\sqrt{n+1}}\sinh\hat{\gamma}+\hat{\gamma}\cosh\hat{\gamma}\right)\xi + n\left(\frac{\hat{\gamma}}{\sqrt{n+1}}+1\right)\sinh\hat{\gamma}\xi\right]}{\left(1-\frac{n+1}{n}\bar{\Omega}^2\right)\hat{\gamma}^2\left(\frac{\hat{\gamma}}{\sqrt{n+1}}\sinh\hat{\gamma}+\hat{\gamma}\cosh\hat{\gamma}\right)+3n\left(\frac{\hat{\gamma}}{\sqrt{n+1}}+1\right)(\hat{\gamma}\cosh\hat{\gamma}-\sinh\hat{\gamma})} \tag{61}$$

which, for $\bar{\Omega} = 0$, corresponds to the expression deduced by Kerr [29] for the static case.

For $\bar{\Omega} < \bar{\psi}^{1/2}$, the contact pressure is in phase with the exciting couple. The frequency-dependence of the pressure distribution is shown in Fig. 10, where the quantity $[\bar{q}(\xi)/\mathscr{F}_0/2l]$ is plotted against ξ for $n = 3$, $\bar{\psi} = 4$, $\gamma_2 = 5$ and $(L/l) = 0{\cdot}6$ (only for $\bar{\Omega} < \bar{\psi}^{1/2}$). For $\bar{\Omega} > \bar{\psi}^{1/2}$, $\bar{q}(\xi)$ is defined by a complex quantity which denotes a phase shift along beam with respect to the exciting couple

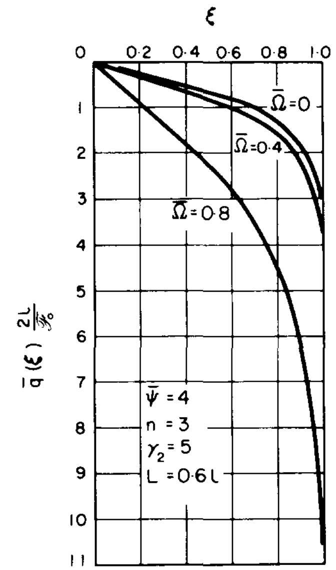

FIG. 10. Contact pressure distribution beneath beam for anti-symmetrical vibrations ($\bar{\Omega} < \bar{\psi}^{1/2}$).

4. CONCLUSIONS

In this theoretical paper a Kerr-type inertial foundation model was developed in order to approximate the dynamic response of a particular kind of vibration isolator manufactured as a carpet with studs on both sides (although it can be also used for other layered deformable supporting media). The analysis was conducted on the simplest two-dimensional problem concerning the steady-state motion of an elastically supported rigid beam. For viscoelastic materials, the springs could be replaced by viscoelastic elements as has been shown by Achenbach and Sun [30] and Kerr [26].

Due to the presence of the shear layer, the beam response is influenced by the outside part of the foundation. The upper spring layer gives a contact-pressure distribution with neither concentrated reactions (as for the Pasternak model) nor infinite values (as for the elastic continuum). The consideration of shear layer inertia is somewhat arbitrary but permits the description of surface wave propagation and variation of pressure distribution with exciting frequency in both phase and magnitude along beam. That is, the proposed model is the simplest one which describes dynamic pressure distributions similar to those experimentally determined by Chae *et al.* [31] for a circular footing on soil (though the static distribution is different). For application to other types of supporting media, the model requires further improvements, especially concerning the reponse in the low-frequency range.

Despite some shortcomings already shown in the paper, the proposed model offers a compromise between mathematical simplicity and accurate representation (at least qualitatively) of the dynamic response of an actual system. Considering only the limit case of a Pasternak-type inertial foundation ($n \to \infty$), the resonant frequencies can be obtained by equating the denominator of equation (26) to zero. Figure 11, in which the mass ratio ψ is plotted against a dimensionless resonant frequency $\omega_{res}(m/k_2)^{1/2}$ seems to explain the experimental results of Arnold *et al.* (Fig. 8(a) in [10]), the curves converging to the same frequency $\omega_{res} = (k_2/m)^{1/2}$ for $\psi = 0$. In Fig. 12, the mass ratio ψ is plotted against a resonant frequency factor $\omega_{res}l/(G/m)^{1/2}$, similar to that generally used for the elastic continuum model. Each curve represents a different value of γ_2 (which is proportional to the length of the beam). The curves are similar to those obtained by Warburton (Fig. 6 in [11]) for an elastic stratum, using the isotropic continuum theory.

A large number of mathematical models, which vary in the degree of sophistication by which they characterize the response of a deformable subgrade to a continuously supported

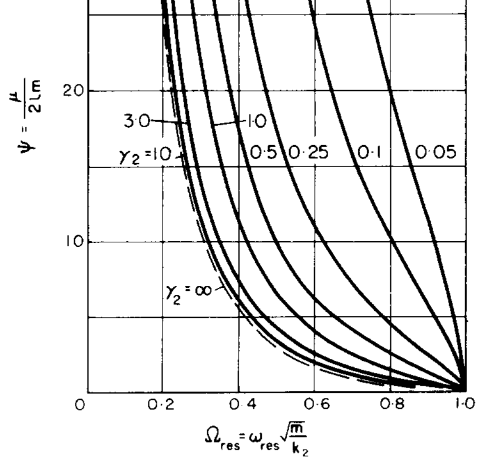

FIG. 11. Mass ratio vs. dimensionless resonant frequency for symmetrical vibrations.

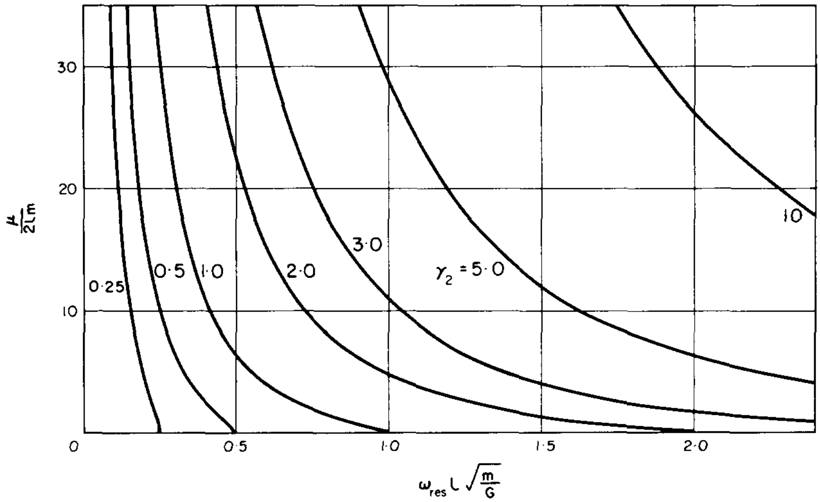

FIG. 12. Mass ratio vs. resonant frequency factor for symmetrical vibrations.

structure, are now available. However, the analytical difficulties encountered in obtaining technically useful solutions and the practical difficulties occurring in determining the corresponding elastic and inertial parameters considerably limit their applicability. In the case of the isotropic elastic half-space, the use of simplified spring-dashpot analogues [32, 33] seems to give a practical solution of the problem. In the case of deformable layers, the theoretical difficulties seem to be overcome by improving the inertial foundation models [28] or by using the finite element technique [34].

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(*Received* 6 *May* 1971; *revised* 26 *January* 1972)

Абстракт—В целью учета зффекта инерции фундамента, обобщается здесь трехпараметрическая безвесовая модель фундамента, предложенная Керром /29/, которая состоит из двух упругих, пружинных слоев, взаимосвязанных упругим срезывающим слоем. Исследуется устойчивое движение жесткой балки, лежащей на поверхности двухмерной модели, учитивая постоянный контакт и распределенную массу срезывающего слоя. Даются большое число диаграммов, касающихся распределения контактного давления под балкой, указывающих его изменение в зависимости от увеличения частоты фазы и размера. Для граничного случая, указывается на качественную сходимость с доступными зкспериментальными данными.